UDC 62-50

DIFFERENTIAL ESCAPE GAMES WITH INFORMATION LAG

PMM Vol. 43, No. 2, 1979, pp. 195-203

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(Received March 6, 1978)

Nonlinear differential escape games with geometric and integral constraints on the controls are examined under the assumption that at each instant t the evader, when constructing his own control, uses the values of the phase vector $z(s)$ and of his opponent's control $u(s)$ for all s such that $s \leqslant t - \tau(z(t))$, where $\tau(z) = \rho(z) / c(z)$. The function $\rho(z)$ tends to zero if the distance between z and the terminal set tends to zero; $c(z)$ is the positive "rate of propagation of information". Lower estimates are obtained for function $c(z)$, for which the fulfilment of the escape conditions in [1 — 3] guarantees the existence of an escape strategy for any initial position.

The motion of a vector z in an n-dimensional Euclidean space R^n is described by the equation

$$dz/dt = f(t, z, u, v), \quad t \in R^1 = (-\infty, +\infty) \tag{1}$$

$$z \in R^n, \quad u \in P, \quad v \in Q$$

where u and v are the controls, P and Q are sets in R^n, and $f(t, z, u, v)$ is a function continuous on $Y = R^1 \times R^n \times P \times Q$. Player U determines the values of parameter u and player V, of parameter v. The admissible controls of players U and V are measurable vector-valued functions $u(t)$ and $v(t)$ satisfying geometric or integral constraints. Controls $u(t)$ and $v(t)$ satisfy geometric constraints if $u(t) \in P$ and $v(t) \in Q$, $t \in R^1$, P and Q are compacta in R^n; controls $u(t)$ and $v(t)$ satisfy integral constraints if

$$\int_{-\infty}^{+\infty} |u(t)|^2 dt \leqslant \rho^2, \quad \int_{-\infty}^{+\infty} |v(t)|^2 dt \leqslant \sigma^2, \quad u(t) \in P, \quad v(t) \in Q, \quad t \in R^1$$

The set $M = \{z \mid \varphi(z) = 0, \ \Phi(z) = 0\}$, where $\varphi(z)$ and $\Phi(z)$ are scalar continuously differentiable functions, is called the terminal set.

We say that the data listed describes a differential game which we call game (1.1) when there are geometric constraints on the controls or game (1.2) when there are integral constraints. We assume that in game (1.2) the set Q contains the null vector of R^n. We denote $R(u, v) = 1 + |u| + |v|$, $P(r) = \{u | u \in P, \ |u| \leqslant r\}$, $Q(r) = \{v | v \in Q, \ |v| \leqslant r\}$, $r_0 = \inf \{r | P(r) = P, \ Q(r) = Q\}$; if P and Q are compacta, $r_0 < +\infty$.

Let the function $f(t, z, u, v)$ satisfy the following conditions:

1) for any $r > 0$ there exists $N(r) < +\infty$ such that for all $(t, z_i, u, v) \in Y$, $|z_i| \leqslant r$, $i = 1, 2$

$$|f(t, z_1, u, v) - f(t, z_2, u, v)| \leqslant N(r)\, R(u, v)\, |z_1 - z_2|;$$

2) a constant $B \geqslant 1$ exists such that for all $(t, z, u, v) \in Y$

$$|f(t, z, u, v)| \leqslant BR(u, v)(1 + |z|)$$

Then for any initial position $(t_0, z_0) \in R^1 \times R^n$ and any admissible controls $u(t)$ and $v(t)$ subject to geometric or integral constraints a unique solution $z(t)$ of Eq. (1) exists in Carathéodory's sense, which is called the motion. From 2) it follows that for any motion $z(t)$ in game (1.1)

$$|z(t) - z(t_0)| \leqslant |t - t_0|\, S_1(z(t_0)) \tag{2}$$

for all $t \in J(t_0) = [t_0 - 1,\ t_0 + 1]$ and for any motion $z(t)$ in game (1.2)

$$|z(t) - z(t_0)| \leqslant |t - t_0|^{1/2}\, S_2(z(t_0)) \tag{3}$$

for all $t \in J(t_0)$, where $S_i(z) = B_i(1 + |z|)$, $B_1 = B(1 + 2r_0)\exp[B(1 + 2r_0)]$ for game (1.1) and $B_2 = B(1 + \rho + \sigma)\exp[B(1 + \rho + \sigma)]$ for game (1.2).

If $X_i(z) = \{y \mid |y - z| \leqslant S_i(z)\}$, $i = 1, 2$, then for any motion $z(t)$ we have $z(t) \in X_i(z(t_0))$ for all $t \in J(t_0)$. The game starts from the initial position $(t_0, z_0) \in R = R^1 \times R^n$, where $z_0 \notin M$; motion $z(t)$ is already determined on the half-open interval $(-\infty, t_0]$ and

$$\sigma^2(t) = \sigma^2 - \int_{-\infty}^{t} |v(t)|^2\, dt, \quad \rho^2(t) = \rho^2 - \int_{-\infty}^{t} |u(t)|^2\, dt$$

T h e i n f o r m a t i v e n e s s c o n d i t i o n. At each instant t player V constructs his own control $v(t)$ by using information on the values of $z(s)$ and $u(s)$ for all $s \leqslant t - \tau(z(t))$, where $\tau(z)$ is a positive function on $R^n \setminus M$.

If for the given initial position we can find a method for constructing an admissible control $v(t)$ in accord with the informativeness condition, then $z(t) \notin M$ for all $t \geqslant t_0$ under any admissible control $u(t)$, then we say that an escape strategy exists for player V from the initial position (t_0, z_0).

Let an operator D associate the function

$$Dh(t, z, u, v) = \frac{\partial h}{\partial t}(t, z) + (\text{grad}_z\, h(t, z) \cdot f(t, z, u, v))$$

with each differentiable function $h(t, z)$ (the argument (t, z) will be denoted by the symbol $(\cdot)$ and the argument (t, z, u, v), by the symbol $[\cdot]$).

C o n d i t i o n 1. Positive integers k, and $l\,(k \leqslant l)$ and scalar functions $h_i(\cdot)$, $g_i[\cdot]$, $i = 0, \ldots, k$, $H_i(\cdot)$, $G_i[\cdot]$, $i = 0, \ldots, l$, continuously differentiable with respect to (t, z) exist such that

$$h_0(\cdot) \equiv \varphi(z), \quad H_0(\cdot) \equiv \Phi(z), \quad g_0[\cdot] \equiv G_0[\cdot] \equiv 0$$

$$g_k[\cdot] = g^1(v) + g^2(t, z, u), \quad G_l[\cdot] = G^1(v) + G^2(t, z, u)$$

$$Dh_i[\cdot] = h_{i+1}(\cdot) + g_{i+1}[\cdot], \quad i = 0, \ldots, k - 1$$

$$DH_i[\cdot] = H_{i+1}(\cdot) + G_{i+1}[\cdot], \quad i = 0, \ldots, l - 1$$

By L we denote a two-dimensional Euclidean space of row-vectors $a = [a^1, a^2]$, where $a^i \in R^1$, $i = 1, 2$; by S_r we denote a sphere of radius r in L : $S_r = \{a \mid a \in L,\ |a| = ((a^1)^2 + (a^2)^2)^{1/2} \leqslant r\}$; by πz we denote the vector $[\varphi(z), \Phi(z)]$. Let $l(\cdot) = [l^1(\cdot), l^2(\cdot)]$, $\Delta(\cdot) = [\Delta^1(\cdot), \Delta^2(\cdot)]$, $\lambda(z)$ be a scalar function.

We denote

$$F[\cdot] = [g_k[\cdot] / k!,\ G_l[\cdot] / l!]$$
$$W_{\varepsilon, r}(\cdot) = \bigcap_{u \in P(r)} \Big(\bigcup_{v \in Q(r+\varepsilon\lambda(z))} F(t, z, u, v) - l(t, z) \Big)$$
$$I_\varepsilon(\cdot) = \bigcup_{\tau \in [-1, 1]} \varepsilon\tau\Delta(\cdot), \quad \mu(z) = \min_{y \in X_2(z)} \lambda(y)$$

C o n d i t i o n 2 . 1. Scalar function $\gamma(t, z)$ and vector-valued functions $l(t, z)$ and $\Delta(t, z)$, all continuously differentiable with respect to (t, z) , and a positive function $\Gamma(z)$, continuous on R^n , exist such that for all $(t, z) \in R$

a) there is fulfilled the inclusion (rough case, $k \leqslant l$)

$$S_{\gamma(\cdot)} \subset W_{0, r_0}(\cdot)$$

or (fine case, $k < l$)

$$I_1(\cdot) \subset W_{0, r_0}(\cdot);$$

b) $\Gamma(z) \leqslant \min \{\gamma(\cdot), \Delta^1(\cdot), \Delta^2(\cdot)\}$

C o n d i t i o n 2 . 2. Scalar function $\gamma(t, z)$ and vector-valued functions $l(t, z)$ and $\Delta(t, z)$, all continuously differentiable with respect to (t, z) , and positive functions $\lambda(z)$ and $\Gamma(z)$, continuous on R^n , exist such that for all $(t, z) \in R$

a) for all $r \geqslant 0$ and $\varepsilon \in (0, 1]$ there is fulfilled the inclusion (rough case, $k \leqslant l$)

$$S_{\varepsilon\gamma(\cdot)} \subset W_{\varepsilon, r}(\cdot)$$

or (fine case, $k < l$)

$$I_\varepsilon(\cdot) \subset W_{\varepsilon, r}(\cdot)$$

b) $\Gamma(z) \leqslant \mu(z) \times \min\{\gamma(\cdot), \Delta^1(\cdot), \Delta^2(\cdot)\}$

C o n d i t i o n 3. A continuous function $m(z) \geqslant 1$, exists such that for all $(t, z, u, v) \in Y$

$$|g_i[\cdot]| \leqslant |\pi z|^{k+1-i} R(u, v)\, m(z), \quad i = 1, \ldots, k-1$$
$$|G_i[\cdot]| \leqslant |\pi z|^{l+1-i} R(u, v)\, m(z), \quad i = 1, \ldots, l-1$$
$$A[\cdot] \leqslant R(u, v)\, m(z)$$
$$(A[\cdot] = |Dh_0[\cdot]| + |Dh_k[\cdot]| + |DH_0[\cdot]| + |DH_l[\cdot]| + |D\gamma[\cdot]| + \sum_{i=1}^{2} (|Dl^i[\cdot]| + |D\Delta^i[\cdot]|) + |g^1(v)| + |g^2(t, z, u)| + |G^1(v)| + |G^2(t, z, u)|)$$

C o n d i t i o n 4 . 1. $\tau(z) = |\pi z| / c(z)$

C o n d i t i o n 4 . 2. $\tau(z) = |\pi z|^\alpha / c(z),\ \alpha > 2l + 3.$

T h e o r e m 1. Let Conditions 1, 2. 1, 3 and 4. 1 be fulfilled for game (1. 1) and let $c(z) \geqslant c_1(z)$, where $c_1(z)$ is a positive function determined by game (1. 1). Then for any initial position $(t_0, z_0) \in R$ we can construct an escape strategy for player V in accordance with the informativeness condition.

T h e o r e m 2. Let Conditions 1, 2. 2, 3 and 4. 2 be fulfilled for game (1. 2) and let $c(z) \geqslant c_2(z)$, where $c_2(z)$ is a positive function determined by game (1. 2).

Then for any initial position $(t_0, z_0) \in R$ we can construct an escape strategy for player V in accordance with the informativeness condition if $\sigma^2(t_0) > 4\rho^2(t_0 - {}^1/_2)$.

Let $\tau \in (0, {}^1/_2]$ and $t^* \in R^1$, assume that motion $z(t)$ has been defined on $(-\infty, t^*]$ and denote $\sigma = t - t^*$, $t_* = t^* - \tau$, $z(t_*) = z_*$, $z(t^*) = z^*$. From Condition 1 it follows that

$$\varphi(z(t)) = \sum_{i=0}^{k} \frac{(\sigma+\tau)^i}{i!} h_i(t_*, z_*) + \tag{4}$$

$$\sum_{i=1}^{k} \int_{t_*}^{t} \frac{(t-s)^{i-1}}{(i-1)!} g_i(s, z(s), u(s), v(s))\, ds +$$

$$\int_{t_*}^{t} \frac{(t-s)^k}{k!} Dh_k(s, z(s), u(s), v(s))\, ds, \quad \sigma \geqslant 0$$

An analogous expression is obtained for $\Phi(z(t))$ by replacing k by l, h by H and g by G. To construct the evasion maneuver we make use of the following representation for $\pi z(t)$:

$$\pi z(t) = T + I + K \tag{5}$$

$$T^1 = \sum_{i=0}^{k} \frac{(\sigma+\tau)^i}{i!} h_i(t_*, z_*) + \frac{\sigma^k}{k!} l^1(t_*, z_*)$$

$$I^1 = \int_{t^*}^{t} \frac{(t-s)^{k-1}}{(k-1)!} (g_k(s-\tau, z(s-\tau), u(s-\tau), v(s)) -$$

$$l^1(s-\tau, z(s-\tau)))\, ds$$

Expressions for T^2 and I^2 are obtained from those for T^1 and I^1 by replacing k by l, h by H and g by G. From Condition 3 and from (2) and (3) it follows that for any motion $z(t)$ with $t \in J(t_0)$

$$|\pi z(t) - \pi z(t_0)| \leqslant |t - t_0|\, P_1(z(t_0)) \text{ for game } (1.1) \tag{6}$$

$$|\pi z(t) - \pi z(t_0)| \leqslant |t - t_0|^{1/\alpha} P_2(z(t_0)) \text{ for game } (1.2)$$

$$(P_i(z) = B_i \max_{y \in X_i(z)} m(y),\ i = 1, 2,\ P_i(z)$$

are functions continuous on R^n.

By $z[t]$ we denote the vector $z(t - \tau(z(t)))$ which, according to the informativeness condition, is known to player V at instant t. Assuming that Condition $(4.i)$ is filfilled for game $(1.i)$, $i = 1, 2$, and that $c(z) \geqslant \max\{2|\pi z|, 2P_1(z)\}$ in game (1.1) or $c(z) \geqslant \max\{2|\pi z|^\alpha, (2|\pi z|^{2\alpha-1} P_2(z))^{1/2}\}$ in game (1.2) and using (6), we get that $\tau(z(t)) \leqslant {}^1/_2$ and ${}^1/_2|\pi z(t)| \leqslant |\pi z[t]| \leqslant {}^3/_2 |\pi z(t)|$ for any motion $z(t)$.

P r o o f o f T h e o r e m 1. For any $\delta \in (0, {}^1/_2]$, $w \in S_1$ and $t_* \in R^1$ we denote by $V_1(\omega)$ $(\omega = (t_*, \delta, w))$ the operator $V_1\colon [t_*, t_* + \delta] \times R^n \times P \to Q$ which associates with each triple (t, z, u) the lexicographic minimum of vectors v from set $\Omega_1(\omega; t, z, u)^*$ such that $F(t, z, u, v) = \gamma(t, z)\, w$ in the rough case and $F(t, z, u, v) = (w^1 + w^2 (t - t_*)/\delta)\, \Delta(t, z)$ in the fine case. From Condition 2.1 it follows that set $\Omega_1(\omega; t, z, u)$ is nonempty for $t \in [t_*, t_* + \delta]$; the control $v(t) = V_1(\omega; t, z(t), u(t))$ is admissible for any absolutely continuous function $z(t)$ and admissible control $u(t)$. We denote (c_1 and N_1 are some constants)

$$n(z) = \min\{1, |\pi z|\}, \quad m_1(z) = N_1 S_1(z) P_1(z)\,(1 + P_1^{l+1}(z))$$

$$\theta_1(z) = c_1 \min\left\{\frac{1}{2}, \frac{\Gamma(z)}{m_1(z)}\right\}, \quad \theta_{1*}(z) = \min_{y \in X_1(z)} \theta_1(y)$$

$$P_{1*}(z) = \max_{y \in X_2(z)} P_1(y)$$

L e m m a 1. If Conditions 1, 2.1 and 3 are fulfilled for game (1.1), then constants $c_1 \in (0,1]$ and $N_1 \geqslant 1$ exist such that for any motion $z(t)$ defined on $(-\infty, t^*]$, any $\sigma^* \in (0, \theta_{1*}(z^*)]$ and any $\tau \in [0, \theta_1(z_*)\sigma^*]$ there exists a vector $w_* \in S_1$ depending on z_* and τ, such that the control $v(t) = V_1(\omega_*; t - \tau, z(t - \tau), u(t - \tau))$, $t \in [t^*, t^* + \theta_1(z_*)]$ $(\omega_* = (t_*, \theta_1(z_*), w_*))$, ensures the fulfilment of the estimate

$$|\pi z(t)| \geqslant c_1 \Gamma(z_*)\,(t - t^*)^{l+1}, \quad t \in [t^* + \sigma^*, t^* + \theta_1(z_*)] \tag{7}$$

The following representation holds under the indicated choice of control $v(t)$:

$$\pi z(t)\, A(\sigma) = T A(\sigma) + w + dA(\sigma) \tag{8}$$

where $A(\sigma)$ is the matrix inverse to matrix $\operatorname{diag}\{a\sigma^k, a\sigma^l\}$ in the rough case or to matrix

$$\begin{Vmatrix} a_1\sigma^k & a_1\alpha_k\sigma^{k+1} \\ a_2\sigma^l & a_2\alpha_l\sigma^{l+1} \end{Vmatrix}$$

in the fine case, and $a = \gamma(t_*, z_*)$, $a_i = \Delta^i(t_*, z_*)$, $i = 1, 2$, $\alpha_k = (k/(k+1))\theta_1(z_*)$. Using (2) and Condition 3 we obtain the estimates

$$|d^1| \leqslant R_1(k, \sigma, \tau)\, m_1(z_*), \quad |d^2| \leqslant R_1(l, \sigma, \tau)\, m_1(z_*) \tag{9}$$

$$\sigma \in [\sigma^*, \theta_1(z_*)], \quad \tau \in [0, \theta_1(z_*)\,\sigma^*]$$

$$R_1(k, \sigma, \tau) = \sigma^{k+1} + \sigma^{k-1}\tau + \tau^k$$

for the components of vector $d = [d^1, d^2]$. The validity of Lemma 1 follows from Pontriagin's lemma [1], representation (8) and estimates (9).

Let us describe the escape strategy of player V from the initial position $(t_0, z_0) \in R$. We set

$$\delta_1(z) = \min\{1, {}^1/_2\, c_1 \min_{y \in X_1(z)} \Gamma(y)\, \theta_1^{l+1}(y)\}$$

$$\sigma_1(z) = \frac{\delta_1(z)}{4P_{1*}(z)}$$

$$\chi_1(z) = \min_{y \in X_1(z)} \min\{{}^1/_2\, \delta_1(y), c_1 \Gamma(y)\, \sigma_1^{l+1}(y)\}$$

We define a sequence of instants $t_i < t_i^*$, $i = 1, 2, \ldots$

$$|\pi z[t_i]| = n(z^\circ)\ \delta_1(z[t_i])$$

$$|\pi z[t]| > n(z^\circ)\ \delta_1(z[t]), \quad t \in [t_{i-1}^*, t_i)$$

$$t_i^* = \min\{t_i + \theta_1(z(t_i - \tau_i)), t_i'\}$$

$$|\pi z\,[t_i']| = 6n\,(z^\circ),\ |\pi z\,[t]| < 6n\,(z^\circ),\ t \in [t_i,\ t_i')$$
$$t_0{}^* = t_0,\quad z^\circ = z\,[t_0]$$

On the interval $[t_i,\ t_i{}^*]$ player V applies the special control

$$v\,(t) = V_1\,(\omega_i;\ t - \tau_i,\ z\,(t - \tau_i),\ u\,(t - \tau_i))$$
$$(\tau_i = n\,(z^\circ)\,\theta_{1*}\,(z\,[t_i])\ \sigma_1\,(z\,[t_i])$$
$$\omega_i = (t_i - \tau_i,\ \theta_1\,(z(t_i - \tau_i)),\ w_i)$$
$$z^i = z\,(t_i - \tau_i)$$

which according to Lemma 1 ensures the estimate

$$|\pi z\,(t)| \geqslant c_1 \Gamma\,(z^i)\,(t - t_i)^{l+1},\ t \in [t_i + n\,(z^\circ)\ \sigma_1\ (z\ [t_i]),\ t_i{}^*] \tag{10}$$

Using the first estimate in (6) and (10), we obtain

$$|\pi z\,(t)| \geqslant {}^1/_4\ |\pi z[t_i]|,\ t \in [t_i,\ t_i + n\,(z^\circ)\ \sigma_1\ (z\ [t_i])]$$
$$|\pi z(t)| \geqslant c_1 \Gamma(z^i)\,(n\,(z^\circ)\ \sigma_1\ (z\ [t_i]))^{l+1},\quad t \in [t_i + n\,(z^\circ)\,\sigma_1\,(z[t_i]),\ t_i{}^*]$$

Since

$$|\pi z\ [t_i{}^*]| > \min\,\{4n\,(z^\circ),\ {}^1/_2 c_1\,\Gamma\,(z^i)\ \theta_1{}^{l+1}\ (z^i)\} > n\,(z^\circ)\ \delta_1\ (z\ [t_i{}^*])$$

we have that $t_i{}^* < t_{i+1}$ for all $i \geqslant 1$. On the interval $[t_i{}^*,\ t_{i+1})$ player V applies an arbitrary admissible control. We denote

$$c_1(z) = \max\left\{2\,|\,\pi z\,|,\ 2P_1(z),\ \max_{y \in X_1(z)} \frac{12 P_{1*}(y)}{\delta_1(y)\,\theta_{1*}(y)}\right\}$$

Then, if $c\,(z) \geqslant c_1\,(z)$, we have

$$\tau\,(z\,(t)) \leqslant \frac{2\,|\,\pi z\,[t]\,|}{c\,(z\,(t))} \leqslant n\,(z^\circ)\,\theta_{1*}\,(z\,[t_i])\,\sigma_1\,(z\,[t_i]) = \tau_i$$
$$t \in [t_i,\ t_i{}^*]$$

Consequently, player V's control satisfies the informativeness condition. Thus, for the initial position $(t_0,\ z_0) \in R$ we have constructed a strategy that is in accord with the informativeness condition and ensures the fulfilment of the inequality

$$|\pi z\,(t)| \geqslant (n\,(z^0))^{l+1}\,\chi_1\,(z\,(t)),\quad t \geqslant t_0$$

P r o o f o f T h e o r e m 2. For any $\varepsilon \in (0,1]$, $\delta \in (0,\ {}^1/_2]$ and $w \in S_1$ we denote by $V_2\,(\nu)$ $(\nu = (t_*,\ z_*,\ \varepsilon,\ \delta,\ w))$ the operator which associates with each triple $(t,\ z,\ u) \in D\,(\nu) = [t_*,\ t_* + \delta] \times R^n \times P$ the lexicographic minimum of vectors v from the set $\Omega_2\,(\nu;\ t,\ z,\ u)$ such that $F\,(t,\ z,\ u,\ v) = \varepsilon\mu\,(z_*)\,\gamma\,(t,\ z)\,w$ in the rough case and $F(t,\ z,\ u,\ v) = \varepsilon\mu\,(z_*)\,(w^1 + w^2\,(t - t_*)^{1/2}/\,\delta^{1/2})\,\Delta\,(t,\ z)$ in the fine case, $|v| \leqslant |u| + \varepsilon$. From Condition 2.2 it follows that set $\Omega_2\,(\nu;\ t,\ z,\ u)$ is nonempty for all $(t,\ z,\ u) \in D\,(\nu)$; for any absolutely continuous function $z\,(t)$ $(z\,(t_*) = z_*)$ and for a measurable vector-valued function $u\,(t) \in P$ the vector-valued function $v\,(t) = V_2\,(\nu;\ t,\ z\,(t),\ u\,(t))$ is measurable and satisfies the inequality

$$|v\,(t)\,| \leqslant |u\,(t)| + \varepsilon \tag{11}$$

We introduce the notation (c_2 and N_2 are some constants)

$$m_2\,(z) = N_2 S_2(z) P_2\,(z)\ \ (1 + P_2{}^{l+1}(z))$$

$$\theta_2(z) = c_2 \min\left\{\frac{1}{2}\left(\frac{\Gamma(z)}{m_2(z)}\right)^2\right\}, \quad \theta_{2*}(z) = \min_{y \in X_2(z)} \theta_2(y)$$

$$P_{2*}(z) = \max_{y \in X_2(z)} P_2(y)$$

L e m m a 2. If Conditions 1, 2.2 and 3 are fulfilled for game (1, 2), then constants $c_2 \in (0, 1]$ and $N_2 \geqslant 1$, determined by game (1.2), exist such that for any motion $z(t)$ defined on $(-\infty, t^*]$, any $\varepsilon \in (0, 1]$, any $\sigma^* \in (0, \theta_{2*}(z^*)]$, and any $\tau \in (0, (\varepsilon^2\theta_2(z_*)\sigma^*)^2]$ there exists a vector $w_* \in S_1$ depending on z_*, ε and τ, such that the control $v(t) = V_2(\nu_*;\ t-\tau,\ z(t-\tau),\ u(t-\tau))$, $t \in [t^*, t^* + \varepsilon^2\theta_2(z_*)]$ $(\nu_* = (t_*, z_*, \varepsilon, \varepsilon^2\theta_2(z_*), w_*))$ ensures the fulfilment of the inequality

$$|\pi z(t)| \geqslant \varepsilon c_2 \Gamma(z_*)(t - t^*)^{l+1/2}, \quad t \in [t^* + \sigma^*,\ t^* + \varepsilon^2\theta_2(z_*)] \tag{12}$$

Representation (8) holds under the choice of control $v(t)$ indicated in the lemma, where matrix $A(\sigma)$ is inverse to the matrix $\operatorname{diag}\{b\sigma^k, b\sigma^l\}$ in the rough case or to the matrix

$$\begin{Vmatrix} b_1\sigma^k & b_1\beta_k\sigma^{k+1/2} \\ b_2\sigma^l & b_2\beta_l\sigma^{l+1/2} \end{Vmatrix}$$

in the fine case, and

$$b = \varepsilon\mu(z_*)\gamma(t_*, z_*), \quad b_i = \varepsilon\mu(z_*)\Delta^i(t_*, z_*), \quad i = 1,2$$

$$\beta_k = k(\theta_2(z_*))^{-1/2}\int_0^1 (1-s)^{k-1}s^{1/2}\,ds$$

Using (3) and Condition 3 we obtain the estimates

$$|d^1| \leqslant R_2(k, \sigma, \tau)\, m_2(z_*), \quad |d^2| \leqslant R_2(l, \sigma, \tau)\, m_2(z_*) \tag{13}$$

$$\sigma \in (\sigma^*, \varepsilon^2\theta_2(z_*)], \quad \tau \in (0, (\varepsilon^2\theta_2(z_*)\sigma^*)^2]$$

$$R_2(k, \sigma, \tau) = \sigma^{k+1/2} + \sigma^{k-1}\tau^{1/2} + \sigma^{k-3/2} + \tau^{k-1/2}$$

for the components of vector $d = [d^1, d^2]$. The assertion of Lemma 2 follows from representation (8), estimates (13) and Pontriagin's lemma [1].

We denote

$$\delta_2(z) = \min\{1, {}^1/_2 c_2 \min_{y \in X_2(z)} \Gamma(y)\theta_2^{l+1/2}(y)\}$$

$$\sigma_2(z) = \left(\frac{\delta_2(z)}{4P_2(z)}\right)^2$$

$$\chi_2(z) = \min_{y \in X_2(z)} \min\{{}^1/_2\delta_2(y),\ c_2\Gamma(y)\sigma_2^{l+1/2}(y)\}$$

We set

$$\varepsilon_i = \varepsilon_0 i^{-\beta}, \quad i = 1, 2, \ldots, \quad \zeta(s) = \sum_{i=1}^{\infty} i^{-s}$$

$$\varepsilon_0 = \min\left\{1,\ \left(\frac{\sigma^2(t_0) - 4\rho^2(t_0 - 1/2)}{2\zeta(4\beta)}\right)^{1/4}\right\}, \quad \beta = \min\left\{\frac{1}{2},\ \frac{\alpha}{4(2l+3)}\right\}$$

From Condition 4.2 it follows that $4\beta > 1$, i.e., $\zeta(4\beta) < +\infty$. Let us define the intervals $[t_i, t_i^*]$ of active and $[t_i^*, t_{i+1})$ of passive behavior of player V.

For each $i = 1, 2, \ldots$ we define the instant t_i as follows:

$$|\pi z [t_i]| = \varepsilon_i^{2(l+1)} n (z^\circ)\ \delta_2 (z [t_i])$$

$$|\pi z [t]| > \varepsilon_i^{2(l+1)}\ n (z^\circ)\ \delta_2 (z [t]),\ t \in [t_{i-1}^*,\ t_i)$$

while the instant $t_i^* = \min \{t_i + \varepsilon_i^2 \theta_2 (z (t_i - \tau_i)),\ t_i'\}$, where τ_i is defined below and t_i' is such that

$$|\pi z [t_i']| = {}^9/_2\ \varepsilon_i^{1/(2\beta)}\ (n (z^\circ))^{2/\alpha}$$

$$|\pi z [t]| < {}^9/_2\ \varepsilon_i^{1/(2\beta)}\ (n (z^\circ))^{2/\alpha},\ t \in [t_i,\ t_i')$$

From the definition of instants t_i and t_i^* and from the second estimate in (6) it follows that

$$t_i^* - t_i \geqslant \varepsilon_i^{1/\beta}\ (n (z^\circ))^2\ \sigma_2\ (z [t_i]) \tag{14}$$

On the interval $[t_i, t_i^*]$ player V applies the special control

$$v (t) = V_2 (\nu_i;\ t - \tau_i,\ z (t - \tau_i),\ u (t - \tau_i))$$

$$\tau_i = \varepsilon_i^{4l+6}\ (n (z^\circ)\ \theta_{2*}\ (z [t_i])\ \sigma_2\ (z [t_i]))^2\ K\ (z [t_i])$$

$$\nu_i = (t_i - \tau_i,\ z^i,\ \varepsilon_i,\ \varepsilon_i^2\ \theta_2\ (z^i),\ w_i)$$

$$K (z) = \min \{1,\ \min_{y \in X_2(z)} \sigma_2\ (y)\}$$

From the second estimate in (6) and from Lemma 2 follows the existence of a vector $w_i \in S_1$ such that control $v (t)$ ensures the estimates

$$|\pi z (t)| \geqslant {}^1/_4\ |\ \pi z\ [t_i]|,\quad t \in [t_i,\ t_i + (n (z^\circ))^2\ \sigma_2\ (z\ [t_i])]$$

$$|\pi z (t)| \geqslant \varepsilon_i c_2 \Gamma\ (z^i)\ \ (t - t_i)^{l+1/2},\quad t \in [t_i + (n (z^\circ))^2 \sigma_2 (z[t_i]),\ \ t_i^*]$$

Hence we get that

$$|\pi z (t)| \geqslant \varepsilon_i^{2(l+1)^2}\ (n (z^0))^{2l+1}\ \chi_2\ (z (t)),\ t \in [t_i,\ t_i^*] \tag{15}$$

and

$$|\pi z\ [t_i^*]| > \varepsilon_{i+1}^{2(l+1)}\ n (z^\circ)\ \delta_2\ (z\ [t_i^*])$$

i.e., $t_i^* < t_{i+1},\ \ i \geqslant 1$.

Player V's control is set identically equal to zero on the interval $[t_i^*,\ t_{i+1})$.

Let $c (z) \geqslant c_2 (z)$,

$$c_2 (z) = \max \left\{ 2\, |\, \pi z\, |^\alpha,\quad (2\, |\, \pi z\, |^{2\alpha-1} P_2 (z))^{1/2},\ \max_{y \in X_2(z)} \frac{6^{\alpha+1} P_{2*} (y)}{K\ (y)\ (\delta_2\ (y)\ \theta_{2*}\ (y))^2} \right\}$$

Then $\tau\ (z\ (t)) \leqslant \tau_i,\ t \in [t_i,\ t_i^*\ ]$. Consequently, the control $v\ (t)$ constructed satisfies the informativeness condition on interval $[t_i,\ t_i^*]$. From estimate (11) follows

$$\int_{t_i}^{t_i^*} |\, v\, (t)\, |^2\, dt \leqslant 2 \int_{t_i}^{t_i^*} |\, u\, (t - \tau_i)\, |^2 + 2\varepsilon_i^4$$

For all $i = 2, 3, \ldots$ we have $t_{i-1} \leqslant t_i - \tau_i$, because if $t_i - t_{i-1} \leqslant {}^1/_2$,

$$\tau_i \leqslant \varepsilon_i^{4l+6}\ (n\ (z^\circ))^2\ \ K\ (z\ \ [t_i]) \leqslant \varepsilon_{i-1}^{1/\beta}\ (n\ (z^\circ))^2\ \ \sigma_2\ \ (z\ [t_{i-1}])$$

consequently, $\tau_i \leqslant t_{i-1}^* - t_{i-1} \leqslant t_i - t_{i-1}$; since $\tau_i \leqslant 1/2$, $t_{i-1} \leqslant t_i - \tau_i$ for all $i \geqslant 2$. Hence

$$\int_{t_0}^{\infty} |v(t)|^2\, dt \leqslant 2 \left(\sum_{i=2}^{\infty} \int_{t_{i-1}}^{t_{i+1}} |u(t)|^2 dt + \int_{t_1-\tau_1}^{t_1^*-\tau_1} |u(t)|^2 dt \right) + 2\varepsilon_0^4 \zeta(4\beta) \leqslant \sigma^2(t_0)$$

It was shown above that $t_i > t_{i-1}^*$ for all $i \geqslant 1$; from (14) it follows that $t_i \to \infty$ as $i \to \infty$. The escape strategy constructed in accord with the informativeness condition guarantees estimate (15) for all $t \in [t_i, t_{i+1}]$, $i \geqslant 1$. Theorem 2 is proved.

REFERENCES

1. Pontriagin, L. S., Linear differential game of escape. Tr. Mat. Inst. im. V. A. Steklov, Vol. 112, 1971.

2. Gusiatnikov, P. B., Escape and l-escape in a many-person differential game. Dokl. Akad. Nauk SSSR, Vol. 232, No. 3, 1977.

3. Gusiatnikov, P. B. and Mezentsev, A. V., Differential escape game with integral constraints on the players' controls. Dokl. Akad. Nauk SSSR, Vol. 232, No. 4, 1977.

Translated by N. H. C.